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SWIRLING CONVECTIVE JETS: ASYMPTOTIC ANALYSIS

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Results of solution of the problem of a swirling convective jet propagating over a point heat source on the basis of the equations defining a laminar boundary layer in the Boussinesq approximation are presented. The cases of steady circulation of the azimuthal velocity component of the fluid flow at infinity and the angular momentum of the fluid relative to the axis of the jet were considered. The two variants of the problem were asymptotically analyzed, and corresponding boundary problems were formulated for the principal terms of the expansion of desired functions. The dependences of the velocity and temperature fields on the Prandtl number were determined. Analytical solutions and detailed tables of numerical-analysis data are given.

Keywords: *steady swirling jets, free convective heat exchange, laminar regime, mathematical simulation.*

Introduction. The flows in the swirling jets propagating in thermal-power plants are of fundamental importance for the convective heat transfer in these plants. A large number of analytical and numerical investigations were carried out for the purpose of solving this fundamental problem, even for the case of laminar flows [1–4]. However, if the theory of forced swirling jets can be considered as a fully developed science, the processes of momentum, angular-momentum, and heat transfer in convective swirling jets have been investigated to a lesser extent [5–7]. First, this is explained by the absence of standard analytical methods of solving such problems and the necessity to perform cumbersome calculations in the case where purely numerical schemes are used. Second, a swirling flow induced by a heat source represents a nonlinear combination of a swirling jet and a free convective flow, the consideration of which represents a separate complex problem.

Below are results of mathematical simulation of the development of convective swirling jets on the basis of the equations defining a boundary layer in the Boussinesq approximation. The approach proposed by us represents an efficient and fairly convenient, from the engineering standpoint, combination of analytical and numerical methods that make it possible to obtain a desired expression in the closed form and can be used for investigating the influence of the regime parameters on the main hydrodynamical and thermal characteristics of a jet flow.

Basic Equations. Let us consider an axisymmetric flow caused by a point heat source of intensity Q_0, rotating with a constant angular velocity in an incompressible fluid with a temperature T_∞. We will use a cylindrical coordinate system with velocity-vector components u, v, and w. The origin of coordinates is at the point where the source acts, and the x axis is directed vertically upward. The steady-state motion of the viscous fluid will be described by the system of equations defining a laminar boundary layer in the Boussinesq approximation. All properties of the fluid except the density $\rho = \rho_\infty(1 - \gamma(\Delta T)^2)$ are assumed to be constant. In this case, the main equation defining a vertical jet will take the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial \Delta P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\frac{\partial u}{\partial y}\right) + g\gamma(\Delta T)^2, \quad \frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}(yv) = 0, \quad \frac{w^2}{y} = \frac{1}{\rho}\frac{\partial \Delta P}{\partial y},$$
$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{vw}{y} = \nu\left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{y}\frac{\partial w}{\partial y} - \frac{w}{y^2}\right), \quad u\frac{\partial \Delta T}{\partial x} + v\frac{\partial \Delta T}{\partial y} = \frac{\nu}{\mathrm{Pr}}\left(\frac{\partial^2 \Delta T}{\partial y^2} + \frac{1}{y}\frac{\partial \Delta T}{\partial y}\right). \tag{1}$$

The boundary conditions, to which Eq. (1) satisfies, are merely conditions of symmetry of the flow under which the desired quantities tend asymptotically to the parameters of an undisturbed fluid:

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$$y=0: \quad v=\frac{\partial u}{\partial y}=w=\frac{\partial \Delta T}{\partial y}=0\ ;$$

$$y \to \infty: \quad u \to 0\,, \quad \Delta T \to 0\,, \quad w \to 0\,, \quad \Delta P \to 0\,. \tag{2}$$

The formulation of the problem is completed by determination of the integral invariant (at any $x > 0$ the energy transferred by convection is equal to the energy Q_0 released by the point heat source)

$$Q_0 = 2\pi \int_0^\infty \rho C_p u \Delta T y dy = \text{const}\,. \tag{3}$$

It should be noted that, in the boundary problems of the type being considered, the azimuth velocity-vector component w is closely related to u and ΔT. Therefore, analysis of the problem entails mathematical difficulties associated with reduction of the partial differential equations to the ordinary differential equations with the use of the standard method of separation of variables. In this case, it is profitable to find asymptotic damping laws that would make it possible to most clearly define the main characteristics of the process being investigated, with which this process could be explained and its development could be predicted. The case in point is the search for the solution of the problem at $x \to \infty$, i.e., in the region where the influence of the thermogravitation on the hydrodynamics and the heat exchange is larger by at least an order of magnitude than the influence of the rotation effect, which allows one to disregard (in the first approximation) the term $\partial \Delta P / \partial x$ in the equation of system (1). Then the key moment of the calculation is determination of conditions defining the individual features of a swirling flow. These features can be a finite angular momentum relative to the axis of the jet

$$L_0 = 2\pi \int_0^\infty \rho u w y^2 dy = \text{const}\,, \tag{4}$$

or a constant circulation of the azimuthal component of the velocity vector at infinity

$$yw \to 2\pi\Gamma_0 = \text{const} \neq 0 \quad \text{at} \quad y \to \infty\,. \tag{5}$$

In the first case, an ordinary swirling jet is simulated, and the second variant is simulation of a free convective flow arising over a heat vortex source under extremum natural conditions, such as convective vortices, sandstorms, and tornadoes [8]. In the case where conditions (5) are fulfilled, the distribution of w in the outer region corresponds to a free vortex ($w \sim 1/y$). Now we will consider the solution of the problems set.

Swirling Convective Jet. In accordance with the ordinary procedure of analysis, on the basis of expressions (1)–(4), we will write expressions for the distributions of u, w, and ΔT in the field of a swirling convective jet:

$$u = \left(\frac{g\gamma Q_0^2}{8\pi^2\mu^2 C_p^2}\right)^{1/2} f'\, x^{-1/2}\,, \quad \eta = \left(\frac{g\gamma Q_0^2}{2\pi^2\mu^2 C_p^2 \nu^2}\right)^{1/2} \frac{y^2}{4} x^{-3/2}\,,$$

$$w = \frac{L_0}{4\pi\mu}\left(\frac{g\gamma Q_0^2}{2\pi^2\mu^2 C_p^2 \nu^2}\right)^{1/4} b x^{-7/4}\,, \quad \Delta T = \frac{Q_0}{2\pi\mu C_p} h x^{-1}\,. \tag{6}$$

Then the following equations will be true for the unknown functions $f(\eta)$, $b(\eta)$, and $h(\eta)$ (the prime denotes the derivative with respect to η)

$$(\eta f'')' + \frac{1}{2} f f'' + \frac{1}{4} f'^2 + h^2 = 0\,, \quad \frac{1}{\Pr}(\eta h')' + \frac{1}{2}(fh)' = 0\,, \quad (\eta b')' + \frac{1}{2}(fb)' - \frac{1}{4\eta}(1-f)\, b = 0$$

with the boundary conditions

$$f(0)=0\,,\ \lim_{\eta\to 0}\sqrt{\eta}\,f''=0\quad \lim_{\eta\to 0}\sqrt{\eta}\,h'=0\,,\ b(0)=0\,,\ b(\infty)=0\,,\ f'(\infty)=0\,,\ h(\infty)=0 \tag{7}$$

and the integral conditions

$$\int_0^\infty f'hd\eta=1\,,\ \int_0^\infty f'b\eta^{1/2}d\eta=1\,.$$

For the two special cases Pr = 5/8 and Pr = 1, system (7) admits the solution in quadratures

$$\begin{aligned}&\mathrm{Pr}=5/8:\ f=\frac{24}{5}\frac{\alpha\eta}{1+\alpha\eta}\,,\ h=\frac{4\sqrt{6}}{5}\frac{\alpha}{(1+\alpha\eta)^{3/2}}\,,\ b=\frac{17\sqrt{\alpha}}{10}\frac{\sqrt{\alpha\eta}}{(1+\alpha\eta)^{12/5}}\,,\ \alpha=\frac{125\sqrt{6}}{1152}\,;\\&\mathrm{Pr}=1:\ f=\frac{4\alpha\eta}{1+\alpha\eta}\,,\ h=\frac{2\alpha}{(1+\alpha\eta)^2}\,,\ b=\frac{3\sqrt{\alpha}}{2}\frac{\sqrt{\alpha\eta}}{(1+\alpha\eta)^2}\,,\ \alpha=\frac{3}{8}\,.\end{aligned} \tag{8}$$

The exact analytical solutions (8) allow one to analyze the main characteristics of the phenomenon being investigated and the influence of the regime parameters on the characteristics of the flow in the jet in the explicit form. The quantity $f(0, \mathrm{Pr})$ gives an estimate of the change in the axial values of the velocity u, $h(0, \mathrm{Pr})$ accounts for the change in the temperature ΔT, $f(\infty, \mathrm{Pr})$ defines the mass flow rate m in the jet, and $b_m(\mathrm{Pr})$ gives an idea of the maximum value of the rotational velocity w. At Pr = 5/8, b_m = 0.25664 and η = 0.990. In the case where Pr = 1, b_m = 0.29831 and η = 0.889. It follows herefrom that, when the Prandtl number increases, the value of b increases and its maximum shifts to the axis of the jet flow. The intensity of the rotational motion of the flow is characterized by the parameter $d'(0, \mathrm{Pr})$. Since $d(\eta) = \eta^{1/2}b(\eta)$, $d'(0, 5/8) = \frac{425\sqrt{6}}{2304} = 0.4518373$ and $d'(0, 1) = \frac{9}{16} = 0.5625$.

To obtain a more complete pattern of the phenomenon being investigated, we also numerically solved system (7) that features a singularity at η = 0, which creates serious difficulties for numerical integration of the problem and leads to the appearance of errors in the calculation data. Then, taking into account the peculiar features of free convective flows (at Pr >> 1, the desired solutions have rapidly and slowly changing components at a time), we numerically solved the problem by a more efficient method with the use of the somewhat rearranged equalities (the prime denotes the derivative with respect to ζ)

$$\begin{aligned}&F'''-2F''+F'+\frac{1}{2\mathrm{Pr}}\left(FF''-FF'+\frac{1}{2}F'^2\right)+H^2=0\,,\\&H'+\frac{1}{2}FH-H=0\,,\ G'+\frac{1}{2\mathrm{Pr}}FG-G=0\,,\ \int_{-\infty}^{+\infty}F'H\exp(-\zeta)\,d\zeta=1\,,\ \int_{-\infty}^{+\infty}F'Gd\zeta=1\,.\end{aligned} \tag{9}$$

Relations (9) were obtained as a result of the rearrangement of (7) with the use of the transformations

$$f(\eta)=\mathrm{Pr}^{-1}F(\zeta)\,,\ h(\eta)=\mathrm{Pr}\exp(-\zeta)\,H(\zeta)\,,\ d(\eta)=\mathrm{Pr}\,G(\zeta)\,,\ \zeta=\ln \mathrm{Pr}^{3/2}\eta\,. \tag{10}$$

Relations (9) differ from relations (7) in that they are free of singularity. The second feature is that the new variables (10) allow one to directly determine the large changes in the temperature of the boundary layer arising as a result of an increase in the Prandtl number. The solutions for the functions $F(\zeta)$, $H(\zeta)$, and $G(\zeta)$ were found with the use of the Runge–Kutta integration scheme of the fourth order of accuracy in combination

TABLE 1. Dependence of $f'(0, \mathrm{Pr})$, $h(0, \mathrm{Pr})$, and $f(\infty, \mathrm{Pr})$ on the Pr Number

Pr	$f'(0, \mathrm{Pr})$	$h(0, \mathrm{Pr})$	$f(\infty, \mathrm{Pr})$
0.625	1.2757760	0.5208333	4.800
0.7	1.3249837	0.5686250	4.521
0.72	1.3376071	0.5811480	4.462
0.73	1.3438506	0.5873787	4.435
1	1.5000000	0.7500000	4.000
2	1.9818211	1.3172338	3.669
5	3.0780385	2.9620627	3.570
6.7	3.5832460	3.8841086	3.559
7	3.6668355	4.0464224	3.558
10	4.4347944	5.6648516	3.550
11.4	4.7598014	6.4178381	3.548
50	10.7272347	26.9692196	3.538

TABLE 2. Dependence of $d'(0, \mathrm{Pr})$ on the Pr Number

Pr	$\Gamma(\infty) = 0$ $d'(0, \mathrm{Pr})$	$\Gamma(\infty) = \mathrm{const} \neq 0$ $d'(0, \mathrm{Pr})$
0.625	0.4518373	0.3721014
0.7	0.4780606	0.3732853
0.72	0.4846746	0.3734873
0.73	0.4879234	0.3735757
1	0.5625000	0.3750000
2	0.7202055	0.3922889
5	0.9565998	0.4711615
6.7	1.0523609	0.5113981
7	1.0678419	0.5180922
10	1.2065202	0.5795033
11.4	1.2636265	0.6053132
50	2.2204284	1.0504400

with the Newton–Raphson method to satisfy the conditions at the outer boundary of the boundary layer. The numerical analysis was carried out with "opened eyes" with the use, as fixed points, of the quantities determined by formulas (8). The results of the calculations are presented in Tables 1 and 2. They give complete information on the nature of change in the main velocity and temperature characteristics of the jet flow. The fact that the behavior of the quantities $f'(0, \mathrm{Pr})$, $h(0, \mathrm{Pr})$, and $d'(0, \mathrm{Pr})$ throughout the range of change in the Pr number ($0.625 \le \mathrm{Pr} \le 50$) being considered remains practically unchanged have engaged our attention. In this respect, problem (1)–(4) differs substantially from the analogous mathematical model of swirling convective jets constructed on the assumption that the density of the fluid in such a jet depends linearly on its temperature. In [7], it was establishment that at $\rho = \rho_\infty(1 - \beta\Delta T)$ the dependence $d'(0, \mathrm{Pr})$ is nonmonotone: the rotational motion of the jet increases when the Pr number increases to 3.475 and its intensity decreases with a further increase in Pr. The indicated feature is due to the competition of the two mechanisms of vorticity transfer — viscous diffusion and convection. The result obtained in the present work is not unexpected because, in the case of a square dependence of the density of the fluid on its temperature, the buoyancy forces play a dominant role in the process of jet motion. Returning to the initial formulation of problem (1)–(4), we will analyze the region of suitability of the constructed solutions of (6). The results obtained hold true for the case where

$$\omega << 1\,, \quad \omega = \frac{L_0^2}{8\pi^2\mu^2\nu}\left(\frac{2\pi^2\mu^2C_p^2}{g\gamma Q_0^2}\right)^{1/2} x^{-5/2}\,. \tag{11}$$

Heat Vortex Source. Let us consider a swirling jet motion of fluid, for which condition (5) is fulfilled, and determine the asymptotic properties of the solution of the problem at $x \to \infty$. We will use the regularly varying functions (6) and present w in the form

$$w = \pi\Gamma_0 \left(\frac{g\gamma Q_0^2}{2\pi^2 \mu^2 C_p^2 \nu^2} \right)^{1/4} b(\eta)\, x^{-3/4} . \tag{12}$$

Substitution of (12) into (1) gives the equation

$$(\eta b')' + \frac{1}{2} f b' - \frac{1}{4\eta} (1 - f)\, b = 0 , \tag{13}$$

with the boundary conditions

$$b(0) = 0 , \quad \lim_{\eta \to \infty} \sqrt{\eta}\, b = 1 .$$

We will write Eq. (13) in the form suitable for the further investigation ($d(\eta) = \eta^{1/2} b(\eta)$)

$$\eta d'' + \frac{1}{2} f d' = 0 , \quad d(0) = 0 , \quad d(\infty) = 1 . \tag{14}$$

Integration of (14) gives

$$d(\eta) = c_1 \int_0^{\eta} \exp \left(-\frac{1}{2} \int_0^{\eta} \frac{f}{\eta} d\eta \right) d\eta , \quad c_1 = \left[\int_0^{\eta} \exp \left(-\frac{1}{2} \int_0^{\infty} \frac{f}{\eta} d\eta \right) d\eta \right]^{-1} .$$

Using the analytical solutions for the function f (8), we find

$$\mathrm{Pr} = 5/8 : \quad d = 1 - \frac{1}{(1 + \alpha\eta)^{7/5}} , \quad \alpha = \frac{125\sqrt{6}}{1152} ;$$

$$\mathrm{Pr} = 1 : \quad d = 1 - \frac{1}{1 + \alpha\eta} , \quad \alpha = 3/8 . \tag{15}$$

It follows herefrom that $h'(0, 5/8) = \frac{175\sqrt{6}}{1152} = 0.3721013$ and $d'(0, 1) = 3/8 = 0.3750$. Numerical solution of (14) presents no problems in principle if we go to the new functions $f(\eta) = \mathrm{Pr}^{-1} F(\zeta)$ and $d(\eta) = G(\zeta)$ and to the variable $\zeta = \ln \mathrm{Pr}^{3/2} \eta$:

$$G'' + \frac{1}{2\mathrm{Pr}} F G' - G' = 0 , \quad G(-\infty) = 0 , \quad G(\infty) = 1 . \tag{16}$$

The results of our calculations (Table 2) show that an increase in the Pr number leads to growth in the function $d'(0, \mathrm{Pr})$. Comparison of the structures of the swirling jet flows arising under conditions (4) and (5) points to the fact that the dependences of the intensities of their rotational motion on the Prandtl number are identical in character; however, the rates of increase in these intensities are different. The dependence of the density of the fluid in these flows on its temperature manifests itself as a qualitative change in the hydrodynamic pattern of the jet: if $\rho = \rho_\infty(1 - \beta\Delta T)$, the quantity $d'(0, \mathrm{Pr})$ has a maximum at $\mathrm{Pr}_* = 0.617$. At $\mathrm{Pr} > \mathrm{Pr}_*$ or $\mathrm{Pr} < \mathrm{Pr}_*$ the situation changes: the buoyancy forces suppress the rotation effects [7]. This means that the motion of fluid in swirling convective jets depends, on the one hand, on the Pr number, and on the other on the relation between the density and the temperature of the fluid.

We will analyze the region of suitability of the results obtained in greater detail. The degree of swirling of a jet flow can be characterized by the quantity

$$\omega = \frac{2\pi^2\Gamma_0^2}{\nu}\left(\frac{2\pi^2\mu^2 C_p^2}{g\gamma Q_0^2}\right)^{1/2} x^{-1/2} . \quad (17)$$

A weakly swirled flow corresponds to small values of ω ($\omega < 1$), and at $\omega > 1$ a strongly swirled jet arises. In accordance with (12), $\Delta P \sim x^{-3/2}$ and, therefore, the value of the term with a longitudinal pressure gradient $\frac{1}{\rho}\frac{\partial \Delta P}{\partial x}$ becomes of the order of $x^{-5/2}$. Consequently, this term can be disregarded as compared to the term accounting for the buoyancy forces $g\gamma(\Delta T)^2$ (in the first approximation) in system (1) in the case where $g\gamma(\Delta T)^2 >> \frac{1}{\rho}\frac{\partial \Delta P}{\partial x}$, which is equivalent to the condition $\omega << 1$. In this case, the swirling of the jet weakly influences the change in u, v, and ΔT. At $x \to \infty$, $\omega \to 0$ and the requirement imposed on the value of ω becomes less restrict and, therefore, the range of allowable values of ω becomes wider. Thus the model representation (6), (12) can serve as an efficient instrument in the analysis of completely developed swirling convective jets.

NOTATION

C_p, heat capacity at a constant pressure, J/(kg·K); g, free fall acceleration, m/sec^2; m, mass rate of flow, kg/(m·sec); Pr, Prandtl number; Q_0, flow of excessive heat content, J/sec; P, pressure, kg/(m·sec^2); T, temperature, K; u, v, w, velocity vector components, m/sec; x, y, longitudinal and transverse coordinates, m; β, coefficient of volumetric thermal expansion, 1/K; Γ_0 = const; γ, temperature coefficient, 1/K^2; $\Delta P = (P - P_\infty)$, excessive pressure, kg/(m·sec^2); $\Delta T = (T - T_\infty)$, excessive temperature, K; μ, coefficient of dynamic viscosity, kg/(m·sec); ν, coefficient of kinematic viscosity, m^2/sec; ρ, density, kg/m^3. Subscripts: ∞, surrounding fluid; m, maximum value; 0, constancy of a value; $'$, derivative.

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